Fatigue crack prediction and prognosis using Bayesian probabilistic framework

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Structural fatigue crack prognosis using Bayesian probabilistic framework – Extended Kalman Filter

1. Background (data variability & Bayes model)
2. Kalman Filter
3. Virkler crack propagation test
4. Estimation of crack growth
5. Conclusion
1. Background

Prognosis (or prediction) of fatigue crack growth

- Paris Model
- Forman model
- Willenborg model
- Newman model
- ...
- ...

\[ \frac{da}{dN} = C(\Delta K)^m \]

Uncertainty/variability

Inter-dependent?
1. Background – Prognosis (or prediction) of fatigue crack growth

Randomizing Paris constant $C$ and exponent $m$
Need to randomize $a_0$ as well

$\frac{da}{dN} = C(\Delta K)^m$

Nominal range of values

Crack Prognosis (or prediction)

$\text{Incorporate current measurement } a_k$

Probability distribution of $C$ and $m$

$???
2. Kalman Filter – start from Bayes Theorem

**Past knowledge:**
- Predicted current state $x_k$ given previous observation $y_{k-1}$

$$p(x_k|y_k) = \frac{p(y_k|x_k) \cdot p(x_k|y_{k-1})}{p(y_k|y_{k-1})} = \frac{p(y_k|x_k) \cdot \int p(x_k|x_{k-1}) \cdot p(x_{k-1}|y_{k-1}) \, dx_{k-1}}{p(y_k|y_{k-1})}$$

**Measurement Model** – prob. of meas. given state
- Prior prob. of state given meas. (recursive process)

**State Transition Model** – prob. of new state given previous state
- Normalization factor

**Current experience:**
- Update prediction with current measurement

**State (hidden)**
- Measurement/observation of State

Thomas Bayes (1701-1761)
3. Kalman Filter – Bayes Theorem in state-space domain

State variable (hidden, not directly measurable)

\[ x_k = f(x_{k-1}) + w_k \]

Observation variable (indirectly measurement of state variable)

\[ y_k = h(x_k) + v_k \]

State transition model

Error of State model

Measurement model (relating observation with state)

Error of Measurement model

- **Linear** \( f(x) \) & \( h(x) \) + Gaussian \( w \) & \( v \) = Kalman Filter
- **Non-linear** \( f(x) \) || \( h(x) \) + Gaussian \( w \) & \( v \) = Extended Kalman Filter
- **Highly non-linear** \( f(x) \) || \( h(x) \) + Non-Gaussian \( w \) || \( v \) = Particle Filter

R. E. Kalman (1930-2016)
2. Kalman Filter – graphical description

- Predicted estimate = what Paris says
- Measured = what the Sensor gets
- Updated estimate = what Kalman believes it should be

Initial estimate

Predicted estimate by Crack Growth Model – e.g. Paris Law

Updated estimate (Kalman)

Measurements or observation (Sensor)
3. Kalman Filter — Mathematics: optimal solution to Bayes equation (1) if all the probability distributions are **Gaussian** (2) if state and measurement models are **linear**

\[
\begin{align*}
\hat{x}_{k|k-1} &= F\hat{x}_{k-1|k-1} \\
\hat{P}_{k|k-1} &= F\hat{P}_{k-1|k-1}F^T + Q_k \\
S_k &= H\hat{P}_{k|k-1}H^T + R_k \\
K_k &= \hat{P}_{k|k-1}H^TS_k^{-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - H\hat{x}_{k|k-1}) \\
\hat{P}_{k|k} &= (I - K_kH)\hat{P}_{k|k-1}
\end{align*}
\]

→ **predicted estimate of state**

→ **Variance of predicted estimate**

→ **Variance of residual (or innovation)**

→ **Kalman Gain**

→ **updated estimate of state**

→ **Variance of updated estimate**
3. Kalman Filter – (1) if all the probability distributions are Gaussian
(2) if state and/or measurement models are non-linear

\[
\hat{a}_{k|k-1} = f(a_{k-1|k-1})
\]
\[
\hat{P}_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k
\]
\[
S_k = H\hat{P}_{k|k-1} H^T + R_k
\]
\[
K_k = \hat{P}_{k|k-1} H^T S_k^{-1}
\]

\[
a_{k|k} = \hat{a}_{k|k-1} + K_k (y_k - Ha_{k|k-1}) = (I - K_k H)\hat{a}_{k|k-1} + K_k y_k
\]
\[
P_{k|k} = (I - K_k H)\hat{P}_{k|k-1}
\]

\[
\frac{da}{dN} = C(\Delta K)^m
\]
\[
f(a) = a + dN \cdot B \cdot a^{m/2}
\]
\[
\frac{\partial f}{\partial a} = 1 + dN \cdot B \cdot \frac{m}{2} \cdot a^{(m/2)-1}
\]

Weighted average of prediction & measurement depending on errors in prediction and measurement
2. Kalman Filter – derivation of state propagation model from Paris Law

\[
\frac{da}{dN} = C(\Delta K)^m
\]

\[
\Delta K = \beta \cdot \Delta \sigma \cdot \sqrt{\pi a}
\]

\[
\beta = \frac{1}{\sqrt{\cos\left(\frac{\pi a}{2b}\right)}}
\]

\[
a_k = f(a_{k-1}) + w_k
\]

\[
Q_k = (dN_k \cdot C_{std})^2 \left[ (\Delta \sigma)^2 \pi a_k \cdot \sec\left(\frac{\pi a_k}{2b}\right) \right]^m
\]

\[
A = C \cdot (\Delta \sigma \sqrt{\pi})^m
\]

\[
f(a) = a + dN \cdot A \cdot \left[ \frac{a}{\cos(\pi a/2b)} \right]^{m/2}
\]

Assume Normal distribution

\[
w \sim \mathcal{N}(0, Q)
\]
2. Kalman Filter – measurement model (random walk)

\[ a_k = f(a_{k-1}) + w_k \]  
\[ v_k = h(a_{k-1}) + v_k \]

Non-linear state model & 1st order differentiable, so we can use extended Kalman Filter (EKF)

True crack length plus some random noise gives us the measurement

**What if:**

highly non-linear models + non-Gaussian probability distributions  
= EKF no-good, will need to use somethings like the Particle Filter
3. Virkler crack propagation test

- funded by the US Air Force (USAF) and conducted at Purdue University in 1977
- objective was to investigate the statistical characteristics of metal fatigue behavior, and the variability of fatigue crack propagation properties

- 2024-T3 aluminum alloy, center cracked plate
- nominal properties: $S_y = 350$ MPa, $S_u = 490$ MPa, $S_e = 140$ MPa, $S_{yt} = 285$ MPa, $K_{ic} = 29$ MPa/$\sqrt{m}$
- constant amplitude loading with $\Delta \sigma = 48.26$ MPa, stress ratio of $R = 0.2$, i.e. $\sigma = [12.08, 60.34]$ MPa,
- cyclic load frequency was 20 Hz.
- 68 specimens tested, $a_0 = 9$ mm and $a_N = 50$ mm
3. Virkler Data

$\Pr(a|N)$: by ref [1]

No obvious dominant dist

$\Pr(N|a)$: by ref [1]

Predominantly Log-Normal dist.

- Fastest trajectory #15
- Slowest trajectory #49
- Closest to median #18
3. Virkler Data

\( p(a|N) \):

- No obvious dominant distribution by ref [1] via statistical test
- Histogram more like Normal distribution
4. Estimation of crack growth — estimate of \( m \) and \( C \)

Log-linear equation

\[
\log\left(\frac{da}{dN}\right) = \log(C) + m \cdot \log(\Delta K)
\]

\[
\Delta K = \beta \cdot \Delta\sigma \sqrt{\pi a} = \Delta\sigma \sqrt{\pi a \cdot \sec\left(\frac{\pi a}{2b}\right)}
\]

- optimal \( m = 2.9 \), under minimum total Norm-2 power (Least Square) of fitting errors
- \( C_{\text{med}} = 8.586 \times 10^{-11} \)
- \( C_{\text{std}} = 0.619 \times 10^{-11} \)
4. Estimation of crack growth

\[ R = \left(10 \times 0.00139 \times 10^{-3}\right)^2 = 1.9321 \times 10^{-10} \]

\[ Q_k = (dN_k \cdot C_{std})^2 \left[ (\Delta \sigma)^2 \pi a_k \cdot \sec \left( \frac{\pi a_k}{2b} \right) \right]^m \]

\[ Q_k = (0.619 \times 10^{-11})^2 \cdot (48.26^2 \pi)^{2.9} (dN_k)^2 \left[ a_k \cdot \sec \left( \frac{\pi a_k}{2b} \right) \right]^{2.9} = 6.1648 \times 10^{-12} (dN_k)^2 \left[ a_k \cdot \sec \left( \frac{\pi a_k}{2b} \right) \right]^{2.9} \]

Accurate Measurement error std \approx 1.4 \mu m
4. Estimation of crack growth – Test #15 fastest growth

\[ a_0 = 8.95 \text{ mm} \]
4. Estimation of crack growth – Test #15 fastest growth

Bigger error?
4. Estimation of crack growth – Test #18 median growth
4. Estimation of crack growth – Test #18 median growth
4. Estimation of crack growth – Test #49 slowest growth
4. Estimation of crack growth – Test #49 slowest growth

Much Bigger error than #15 and #18
4. Estimation of crack growth – Test #18 with uniform step size $\Delta a$ ($\Delta N$?)

Estimate the last point at $a = 49.8$mm from
- 49mm (the second last point in Virkler data)
- 49.6mm in the resampled data
4. Distribution of prediction error

Test #18: distributions of prediction errors with two prediction time steps
4. Estimation of crack growth – Test #18 estimate of fracture toughness

An observation:

• for Test #18 the predicted stress intensity factor range ($\Delta K$) is 26.37 MPa$\sqrt{m}$ at the last point (*no fracture* in the tests)
• maximum stress intensity factor is $K_{\text{max}} = 33.2$ MPa$\sqrt{m}$ which has exceeded the nominal fracture toughness $K_{\text{lc}} = 29$ MPa$\sqrt{m}$

• actual fracture toughness $K_{\text{lc}}$ is much higher than the nominal value
• the crack tip might be blunted by cyclic plastic deformation ???
4. Estimation of crack growth – gear tooth cracking
5. Conclusions

- Proposed a scheme for crack growth prognosis using recursive EKF solution – an unsophisticated yet robust prognosis methodology for real-time PHM systems
- Applied to the Virkler & DST Gear Rig data, achieved effective prognosis of fatigue crack growth
  - in terms of the accuracy of prediction and the robustness in dealing with uncertainties in material property parameters used in the Paris Law and with the measurement uncertainty

- Comparing to other methods in the literature, the EKF-based approach is much simpler and more robust, and much more readily adaptable state space formulation for researchers in the fracture mechanics community
Prognosis of Fatigue Crack Growth

Questions ???

Paul Paris
1930-2017

Thomas Bayes
1701-1761

Rodulf E. Kalman
1930-2016

\[ x_k | k = (I - K_k H) \hat{x}_{k | k-1} + K_k y_k \]
Backup slides – Bayes Theorem

\[ p(x_k|y_k) = \frac{p(y_k|x_k) \cdot p(x_k|y_{k-1})}{p(y_k|y_{k-1})} = \frac{p(y_k|x_k) \cdot \int p(x_k|x_{k-1}) \cdot p(x_{k-1}|y_{k-1}) \, dx_{k-1}}{p(y_k|y_{k-1})} \]

\[ p(y_k|y_{k-1}) = \int p(y_k|x_k) \cdot p(x_k|y_{k-1}) \, dx_k \]
Backup slides – State model (combined coefficients)

\[ B = C \left( \beta \cdot \Delta \sigma \cdot \sqrt{\pi} \right)^m \]

\[ A = C \cdot \left( \Delta \sigma \sqrt{\pi} \right)^m \]

**Geometry factor**

\[ \beta = \frac{1}{\sqrt{\cos \left( \frac{\pi}{2} \frac{a}{b} \right)}} \]

**Derivation Eq. (3) in the paper**

\[ f(a_{k-1}) = a_k = a_{k-1} + dN \cdot C \left( \Delta \sigma \cdot \sqrt{\pi} a_{k-1} \sec \left( \frac{\pi a_{k-1}}{2b} \right) \right)^m \]

\[ Y = a + bX, \quad \text{Var}[Y] = b^2 \text{Var}[X] = b^2 \sigma_X^2 \]

\[ Q = \text{Var}[a_k - a_{k-1}] = \left[ dN \cdot C_{\text{std}} \left( \Delta \sigma \cdot \sqrt{\pi} a_{k-1} \sec \left( \frac{\pi a_{k-1}}{2b} \right) \right)^m \right]^2 \]